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DIFFERENTIAL CALCULUS NOTES ON WRAPPED EXPONENTIAL DISTRIBUTION

Hilary I. Okagbue*
Department of Mathematics, Covenant University, Ota, Nigeria

Sheila A. Bishop
Department of Mathematics, Covenant University, Ota, Nigeria

Pelumi E. Oguntunde
Department of Mathematics, Covenant University, Ota, Nigeria

Abiodun A. Opanuga
Department of Mathematics, Covenant University, Ota, Nigeria

Ogbu F. Imaga
Department of Mathematics, Covenant University, Ota, Nigeria

Olasunmbo O. Agboola
Department of Mathematics, Covenant University, Ota, Nigeria.

ABSTRACT

Wrapped probability distributions are used in modeling circular data arising from physical, medical and social sciences. Wrapped exponential distribution is obtained from wrapping exponential distribution in a unit sphere. This work considers the generation of ordinary differential equations whose solutions are the probability functions of wrapped exponential distribution. This will help in understanding the nature of exponential distribution when wrapped in a circle. Different methods can be used in obtaining the solutions to the differential equations generated from the process. Some unique patterns were observed which can channel research activities towards the area. In conclusion, some expressions were obtained that link the probability functions with their respective derivatives.

Keywords: Wrapped exponential distribution, differential calculus, ordinary differential equations, survival function, hazard function, odds function.

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1. INTRODUCTION

The background, overview and motivation are presented in different subsections.

1.1. Wrapped Distribution

In probability and statistics, wrapped probability distributions are continuous probability distributions whose random variable are oriented towards a unit sphere. They are used to model circular data that occur in engineering, geological sciences, physics, meteorology and computing. According to [1], any given continuous probability distribution can be wrapped in the circular form. The wrapped distribution is an attempt to define probability distributions in a unique support. This approach originates on a simple fact that a probability distribution on a unit circle or sphere can be obtained by wrapping a probability distribution supported on the real line. The moments, characteristic functions, moment generating function, probability generating function and entropy of wrapped distributions are similar to their parent (unwrapped) distributions. However, the values (measures of central tendencies, variability, skewness and kurtosis are quite different. Generally, the flexibility nature of some distributions makes them easy candidates for wrapping in order to model circular data that are always encountered in statistical analysis, for example, in the estimation of angular systems [2]. The estimation of the parameters of wrapped distributions are usually computational expensive and complex, especially in the use of the maximum likelihood estimation and method of Chi-squares [3]. Researchers have proposed several wrapped distributions. They include: wrapped normal distribution [4] [5] [6], wrapped Fisher-Bingham distribution [7], wrapped Cauchy distribution [8], wrapped gamma distribution [9], wrapped Lomax distribution [10], wrapped Laplace distributions [11] and wrapped Lognormal, Logistic, Weibull, and extreme-value distributions [12]. It can also be observed that there are instances where a probability distribution can be partially wrapped, the wrapped normal distribution is an example [13].

1.2 Wrapped Exponential Distribution

The wrapped exponential distribution is obtained from the wrapping of exponential distribution over a unit sphere. The wrapping forced the distribution to assume different support, nature and orientation. The details on the distribution can be found in [14] and [15]

1.3. Motivation and Related Works

The aim of the paper is use differential calculus to obtain derivatives and ordinary differential equations of the probability functions of wrapped exponential distribution. This approach has yielded differential equations whose solutions are the respective probability functions of complex distributions, interval bounded probability distributions, simple distributions and convoluted probability distributions. Several differential equations for the probability function of different probability distributions have been proposed as seen in [16-37]. The case for wrapped probability distributions have not be reported to the best of the knowledge of the authors. It can be seen later that wrapping a distribution can influence the nature of the ODE related to it. A clear example can be seen in the case of the hazard function of the exponential whose ODE does not exist because of the constant nature of the hazard function. However, it was shown later that the ODE of the wrapped exponential distribution exists. Overall, the solutions to the differential equations can provide an alternative route in understanding the studied distributions.

2. METHODOLOGY

The differential calculus was used extensively to convert the probability functions to their respective ordinary differential equations. It can be seen as a departure from established methods like estimation and fitting. The ODEs prescribe the orientation, nature and statistical direction of the probability functions. Furthermore, elementary algebraic methods were used to simplify the expressions, thereby linking the functions with their respective derivatives. The differential equations cannot exist outside the domain of the support of the distribution that defined it.

3. RESULTS

Differential calculus was applied to obtain the results. Subsequently some mathematical equations were obtained that link the probability functions together with their derivatives**.**

3.1. Probability Density Function

The pdf of the wrapped exponential distribution defined on a unit circle is given as;

$$f(\theta)=\frac{\lambda e^{-\lambda\theta}}{1-e^{-2\pi\lambda}} \tag{1}$$

The pdf is defined on a support $0 \le \theta < 2\pi$ and a rate parameter $\lambda > 0$.

The derivative of the pdf yields;

$$f'(\theta)=-\frac{\lambda^2 e^{-\lambda\theta}}{1-e^{-2\pi\lambda}} \tag{2}$$

$$f'(\theta)=-\lambda f(\theta) \tag{3}$$

$$\frac{f'(\theta)}{f(\theta)}=-\lambda \tag{4}$$

Equation (4) implies that no mode can be estimated since the distribution is oriented to a circle. The second derivative of the pdf is obtained;

$$f''(\theta)=\frac{\lambda^3 e^{-\lambda\theta}}{1-e^{-2\pi\lambda}} \tag{5}$$

$$f''(\theta)=-\lambda f'(\theta) \tag{6}$$

Again, it can be seen that

$$\frac{f''(\theta)}{f'(\theta)}=-\lambda \tag{7}$$

Subsequently, the following can be obtained;

$$\frac{f'''(\theta)}{f''(\theta)} = -\lambda \tag{8}$$

$$\frac{f^{iv}(\theta)}{f'''(\theta)} = -\lambda \tag{9}$$

Generally, it follows that the divisions of the ODEs follows the same pattern;

$$\frac{f^{(n)}(\theta)}{f^{(n-1)}(\theta)} = -\lambda;\ n \geq 1,\ n \in \mathbb{N} \tag{10}$$

Also, the following can be established using equations (1) and (5);

$$f''(\theta) = \lambda^2 f(\theta) \tag{11}$$

$$\frac{f''(\theta)}{f(\theta)} = \lambda^2 \tag{12}$$

The third derivative of the pdf is obtained as;

$$f'''(\theta) = -\frac{\lambda^4 e^{-\lambda\theta}}{1-e^{-2\pi\lambda}} \tag{13}$$

Linking the pdf and its third derivative to obtain;

$$f'''(\theta) = -\lambda^3 f(\theta) \tag{14}$$

$$\frac{f'''(\theta)}{f(\theta)} = -\lambda^3 \tag{15}$$

Linking the first and its third derivatives of the pdf, that is, using equations (2) and (13) to obtain;

$$f'''(\theta) = \lambda^2 f'(\theta) \tag{16}$$

$$\frac{f'''(\theta)}{f'(\theta)} = \lambda^2 \tag{17}$$

Equations (12) and (17) appear to follow the same pattern. Generally, it can be seen that;

$$\frac{f^{(n)}(\theta)}{f^{(n-2)}(\theta)} = \lambda^2;\ n \geq 2,\ n \in \mathbb{N} \tag{18}$$

Generally, an equation that links equations (10) and (18) is obtained, given as;

$$\frac{f^{(n)}(\theta)}{f^{(n-2)}(\theta)} = \left(\frac{f^{(n)}(\theta)}{f^{(n-1)}(\theta)}\right)^2;\ n \neq 1,2,\ n \in \mathbb{N} \tag{19}$$

The fourth derivative of the pdf is obtained as;

$$f^{iv}(\theta) = \frac{\lambda^5 e^{-\lambda\theta}}{1-e^{-2\pi\lambda}} \tag{20}$$

Linking the first and fourth order differential equations;

$$f^{iv}(\theta) = -\lambda^3 f'(\theta) \tag{21}$$

$$\frac{f^{iv}(\theta)}{f'(\theta)} = -\lambda^3 \tag{22}$$

A generally expression was obtained from equation (22);

$$\frac{f^{(n)}(\theta)}{f^{(n-3)}(\theta)} = -\lambda^3;\ n \geq 3,\ n \in \mathbb{N} \tag{23}$$

Consequently, equation (23) is linked with equations (10) and (18) to obtain two general expressions;

$$\frac{f^{(n)}(\theta)}{f^{(n-3)}(\theta)} = \left(\frac{f^{(n)}(\theta)}{f^{(n-1)}(\theta)}\right)^3;\ n \neq 1,3,\ n \in \mathbb{N} \tag{24}$$

$$\frac{f^{(n)}(\theta)}{f^{(n-3)}(\theta)} = -\lambda\left(\frac{f^{(n)}(\theta)}{f^{(n-2)}(\theta)}\right);\ n \neq 1,2,\ n \in \mathbb{N} \tag{25}$$

$$\frac{f^{(n)}(\theta)}{f^{(n-3)}(\theta)} = \left(\frac{f^{(n)}(\theta)}{f^{(n-1)}(\theta)}\right)\left(\frac{f^{(n)}(\theta)}{f^{(n-2)}(\theta)}\right);\ n \neq 1,2,\ n \in \mathbb{N} \tag{26}$$

3.2. Survival Function

The sf of the wrapped exponential distribution defined on a unit circle is given as;

$$s(\theta) = \frac{e^{-\lambda\theta} - e^{-2\pi\lambda}}{1-e^{-2\pi\lambda}} \tag{27}$$

The sf is defined on a support $0 \leq \theta < 2\pi$ and a rate parameter $\lambda > 0$.

The first derivative of the sf yields the negative times the pdf;

$$s'(\theta) = -\frac{\lambda e^{-\lambda\theta}}{1-e^{-2\pi\lambda}} \tag{28}$$

Higher order derivatives are obtained;

$$s''(\theta) = \frac{\lambda^2 e^{-\lambda\theta}}{1-e^{-2\pi\lambda}} \tag{29}$$

$$s'''(\theta) = -\frac{\lambda^3 e^{-\lambda\theta}}{1-e^{-2\pi\lambda}} \tag{30}$$

$$s^{iv}(\theta) = \frac{\lambda^4 e^{-\lambda\theta}}{1-e^{-2\pi\lambda}} \tag{31}$$

$$s^{v}(\theta) = -\frac{\lambda^5 e^{-\lambda\theta}}{1-e^{-2\pi\lambda}} \tag{32}$$

The odd derivatives are negative while the even ones are positive.

Linking the first and second order differential equations;

$$s''(\theta) = -\lambda s'(\theta) \tag{33}$$

$$\frac{s''(\theta)}{s'(\theta)} = -\lambda \tag{34}$$

Linking the second and third order differential equations;

$$s'''(\theta) = -\lambda s''(\theta) \tag{35}$$

$$\frac{s'''(\theta)}{s''(\theta)} = -\lambda \tag{36}$$

Equations (34) and (36) appear to follow the same pattern. Generally, it can be seen that;

$$\frac{s^{(n)}(\theta)}{s^{(n-1)}(\theta)} = -\lambda;\ n \geq 2,\ n \in \mathbb{N} \tag{37}$$

The derivative starts from n = 2, since the ratio of the derivative and the sf was not does not appear to follow the same pattern.

Linking the first and third order differential equations;

$$s'''(\theta) = \lambda^2 s'(\theta) \tag{38}$$

$$\frac{s'''(\theta)}{s'(\theta)} = \lambda^2 \tag{39}$$

Linking the second and fourth order differential equations;

$$s^{iv}(\theta) = \lambda^2 s''(\theta) \tag{40}$$

$$\frac{s^{iv}(\theta)}{s''(\theta)} = \lambda^2 \tag{41}$$

Equations (39) and (41) appear to follow the same pattern. Generally, it can be seen that;

$$\frac{s^{(n)}(\theta)}{s^{(n-2)}(\theta)} = \lambda^2;\ n \geq 3,\ n \in \mathbb{N} \tag{42}$$

The derivative starts from n = 3, since the ratio of the second derivative and the sf was not does not appear to follow the same pattern

Equations (37) and (42) can be joined in a single expression;

$$\frac{s^{(n)}(\theta)}{s^{(n-2)}(\theta)}=\left(\frac{s^{(n)}(\theta)}{s^{(n-1)}(\theta)}\right)^2;\ n\geq 2,3,\ n\in\mathbb{N} \tag{43}$$

Linking the first and fourth order differential equations;

$$s^{iv}(\theta)=-\lambda^3 s'(\theta) \tag{44}$$

$$\frac{s^{iv}(\theta)}{s'(\theta)}=-\lambda^3 \tag{45}$$

Linking the second and fifth order differential equations;

$$s^{v}(\theta)=-\lambda^3 s''(\theta) \tag{46}$$

$$\frac{s^{v}(\theta)}{s''(\theta)}=-\lambda^3 \tag{47}$$

Equations (45) and (47) appear to follow the same pattern. Generally, it can be seen that;

$$\frac{s^{(n)}(\theta)}{s^{(n-3)}(\theta)}=\lambda^2;\ n\geq 4,\ n\in\mathbb{N} \tag{48}$$

The derivative starts from n = 4, since the ratio of the third derivative and the sf was not does not appear to follow the same pattern.

Five general expressions can be obtained using equations (37), (42) and (48);

$$\frac{s^{(n)}(\theta)}{s^{(n-3)}(\theta)}=\lambda^2\ \frac{s^{(n)}(\theta)}{s^{(n-1)}(\theta)} \tag{49}$$

$$\frac{s^{(n)}(\theta)}{s^{(n-3)}(\theta)}=-\lambda\left(\frac{s^{(n)}(\theta)}{s^{(n-1)}(\theta)}\right)^2 \tag{50}$$

$$\frac{s^{(n)}(\theta)}{s^{(n-3)}(\theta)}=\left(\frac{s^{(n)}(\theta)}{s^{(n-1)}(\theta)}\right)^3 \tag{51}$$

$$\frac{s^{(n)}(\theta)}{s^{(n-3)}(\theta)}=-\lambda\left(\frac{s^{(n)}(\theta)}{s^{(n-2)}(\theta)}\right) \tag{52}$$

$$\frac{s^{(n)}(\theta)}{s^{(n-3)}(\theta)}=\left(\frac{s^{(n)}(\theta)}{s^{(n-1)}(\theta)}\right)\left(\frac{s^{(n)}(\theta)}{s^{(n-2)}(\theta)}\right) \tag{53}$$

$$\frac{s^{(n)}(\theta)}{s^{(n-3)}(\theta)}=-\left(\frac{s^{(n)}(\theta)}{s^{(n-2)}(\theta)}\right)^{3/2} \tag{54}$$

3.3. Hazard Function

The hf of the wrapped exponential distribution defined on a unit circle is given as;

$$h(\theta) = \frac{\lambda e^{-\lambda\theta}}{e^{-\lambda\theta} - e^{-2\pi\lambda}} \tag{55}$$

The hf is defined on a support $0 \leq \theta < 2\pi$ and a rate parameter $\lambda > 0$. The first derivative of the hf yields equation (56);

$$h'(\theta) = \frac{\left(\lambda e^{-\lambda\theta}\right)^2}{(e^{-\lambda\theta} - e^{-2\pi\lambda})^2} - \frac{\lambda^2 e^{-\lambda\theta}}{e^{-\lambda\theta} - e^{-2\pi\lambda}} \tag{56}$$

Factorizing using equation (55);

$$h'(\Theta) = h^2(\Theta) - \lambda h(\Theta) \tag{57}$$

$$\frac{h'(\theta)}{h(\theta)} = h(\theta) - \lambda \tag{58}$$

The second derivative of the hf can be obtained using equation (57);

$$h''(\theta) = 2h(\theta)h'(\theta) - \lambda h'(\theta) \tag{59}$$

$$\frac{h''(\theta)}{h'(\theta)} = 2h(\theta) - \lambda \tag{60}$$

Factorizing the second derivative using the first derivative;

$$h''(\Theta) = 2h^3(\Theta) - 3\lambda h^2(\Theta) + \lambda^2 h(\Theta) \tag{61}$$

$$\frac{h''(\theta)}{h(\theta)} = (h(\theta) - \lambda)(2h(\theta) - \lambda) \tag{62}$$

3.4. Reversed Hazard Function

Reversed hazard function is the ratio of the pdf to the cdf of any given probability distribution. The rhf of the wrapped exponential distribution defined on a unit circle is given as;

$$v(\theta) = \frac{\lambda e^{-\lambda\theta}}{1 - e^{-\lambda\theta}} \tag{63}$$

The rhf is defined on a support $0 \leq \theta < 2\pi$ and a rate parameter $\lambda > 0$. The first derivative of the rhf is given as;

$$v'(\theta) = -\frac{\left(\lambda e^{-\lambda\theta}\right)^2}{(1 - e^{-\lambda\theta})^2} - \frac{\lambda^2 e^{-\lambda\theta}}{1 - e^{-\lambda\theta}} \tag{64}$$

Factorizing using equation (63);

$$v'(\Theta) = -v^2(\Theta) - \lambda v(\Theta) \tag{65}$$

$$v'(\theta) = -v(\theta)(v(\theta) + \lambda) \tag{66}$$

$$\frac{v'(\theta)}{v(\theta)} = -(v(\theta) + \lambda) \tag{67}$$

The second derivative of the rhf can be obtained using equation (65);

$$v''(\theta) = -2v(\theta)v'(\theta) - \lambda v'(\theta) = -v'(\theta)(2v(\theta) + \lambda) \tag{68}$$

$$\frac{v''(\theta)}{v'(\theta)} = -(2v(\theta) + \lambda) \tag{69}$$

In addition, the second derivative can be written as;

$$v''(\theta) = v(\theta)(v(\theta) + \lambda)(2v(\theta) + \lambda) \tag{70}$$

3.5. Odds Function

Odds function is the ratio of the cdf to the sf of any given probability distribution. The odf of the wrapped exponential distribution defined on a unit circle is given as;

$$d(\theta) = \frac{1 - e^{-\lambda\theta}}{e^{-\lambda\theta} - e^{-2\pi\lambda}} \tag{71}$$

The rhf is defined on a support $0 \leq \theta < 2\pi$ and a rate parameter $\lambda > 0$. The first derivative of the rhf is given as;

$$d'(\theta) = \frac{(\lambda e^{-\lambda\theta})(1 - e^{-\lambda\theta})}{(e^{-\lambda\theta} - e^{-2\pi\lambda})^2} + \frac{\lambda e^{-\lambda\theta}}{e^{-\lambda\theta} - e^{-2\pi\lambda}} \tag{72}$$

$$d'(\theta) = \frac{(\lambda e^{-\lambda\theta})d(\theta)}{e^{-\lambda\theta} - e^{-2\pi\lambda}} + \frac{\lambda e^{-\lambda\theta}}{e^{-\lambda\theta} - e^{-2\pi\lambda}} = \frac{\lambda e^{-\lambda\theta}}{e^{-\lambda\theta} - e^{-2\pi\lambda}}(d(\theta) + 1) \tag{73}$$

$$d'(\theta) = h(\theta)(d(\theta) + 1) \tag{74}$$

where $h(\theta)$ is the hazard function.

$$\frac{d'(\theta)}{d(\theta)} = h(\theta) + \frac{h(\theta)}{d(\theta)} \tag{75}$$

$$\frac{d'(\theta)}{d(\theta)} = h(\theta) + v(\theta) \tag{76}$$

where $v(\theta)$ is the reversed hazard function

The second derivative of odf is obtained and is given as;

$$d''(\theta) = h(\theta)d'(\theta) + h'(\theta)d(\theta) + h'(\theta) \tag{77}$$

Some factorizations are done:

$$\frac{d''(\theta)}{d'(\theta)} = h(\theta) + \frac{h'(\theta)d(\theta)}{d'(\theta)} + \frac{h'(\theta)}{d'(\theta)} \tag{78}$$

$$\frac{d''(\theta)}{d(\theta)} = \frac{h(\theta)d'(\theta)}{d(\theta)} + h'(\theta) + \frac{h'(\theta)}{d(\theta)} \tag{79}$$

$$\frac{d''(\theta)}{d(\theta)} = v(\theta)d'(\theta) + h'(\theta) + \frac{h'(\theta)}{d(\theta)} \tag{80}$$

A simplified classes of derivatives can be obtained using the modified product method of differential calculus.

Given the odds function, the derivative can be expressed as its function as shown in the following expressions

$$d'(\theta) = \left[\frac{f(\theta)}{F(\theta)} + \frac{f(\theta)(s(\theta))^{-2}}{(s(\theta))^{-1}}\right]d(\theta) \tag{81}$$

$$d'(\theta) = \left[v(\theta) + h(\theta)\right]d(\theta) \tag{82}$$

$$\frac{d'(\theta)}{d(\theta)} = v(\theta) + h(\theta) \tag{83}$$

Equations (82) and (83) are equations linking the odds function and its derivative, reversed hazard function and hazard function.

Equations (82) is factorized to obtain;

$$d'(\theta) = h(\theta) + h(\theta)d(\theta) \tag{84}$$

Differentiating equation (84) yields equation (77)

3.6. Solution of the Proposed ODEs

Various methods are available in obtaining solutions to ordinary differential equations obtained in this article. Analytical, series, semi-analytical and numerical methods are readily available in obtaining the solutions to the ODE.

4. CONCLUSION

The authors have proposed some forms of ordered differential equations for the probability density function and other related probability functions of wrapped exponential distribution. The result has prescribed alternative nature of the probability functions when the solutions to the ODEs are obtained by any of the aforementioned methods. Some patterns were obtained. Finally, some unique mathematical equations were obtained that link some probability functions with their respective derivatives.

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